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Weighted Posterior Odds: A Data Summary for Decision Making

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Outline

- Bayesian model, loss function for testing a 1-sided hypothesis
  - Bayes rule critical region: weighted posterior odds is the focal point
  - Normal data, linear loss as a special case
  - Numerical examples
- Bayes model, loss function for testing a 1-sided comparison of means
  - Bayes rule critical region: weighted posterior odds is the focal point
  - Normal data, linear loss as a special case
  - Numerical examples
- Linear loss in testing $\Longleftrightarrow$ squared-error loss in estimation connection
- Discussion

Bayesian Model

Data

$$\underline{\boldsymbol{y}} = (y_1, y_2, \dots y_n)$$

$$y_i \overset{iid}{\sim} f(y|\mu, \gamma)$$

Hypothesis Test

$$H: \mu \leq \mu_0$$

$$A: \mu > \mu_0$$

Loss Model

Loss Function $L(a, \mu) = L_a(\mu)$

Decision $a = I\left(\underline{y} \in R\right)$

Critical Region $R = \left\{\underline{\boldsymbol{y}}: \text{accept } A: \mu > \mu_0 \text{ as true}\right\}$

Type 1 Loss $L_1(\mu) = \boldsymbol{k} l(\mu_0, \mu) I(\mu \leq \mu_0)$

Type 2 Loss $L_0(\mu) = \; l(\mu, \mu_0) I(\mu > \mu_0)$

Distance Function $l(u, v) \begin{cases} = 0, u \leq v \\ > 0, u > v \end{cases}$

$\boldsymbol{k}$ = type 1 to type 2 error loss (or seriousness) ratio

Bayes Rule Critical Region

$$\tilde{R} = \left\{ \underline{\boldsymbol{y}} : W\left(\underline{\boldsymbol{y}}, \mu_0\right) > k \right\}$$

where

$$W\left(\underline{\boldsymbol{y}}, \mu_0\right) = \frac{\int_{\mu_0}^{\infty} l(\mu,\mu_0)\pi\left(\mu|\underline{\boldsymbol{y}}\right)d\mu}{\int_{-\infty}^{\mu_0} l(\mu_0,\mu)\pi\left(\mu|\underline{\boldsymbol{y}}\right)d\mu} =$$ **weighted posterior odds of $\boldsymbol{A}$**

$\pi\left(\mu|\underline{\boldsymbol{y}}\right) =$ posterior of $\mu|\underline{y}$

Bayes Rule Critical Region

$$\tilde{R} = \left\{\underline{\boldsymbol{y}}: W\left(\underline{\boldsymbol{y}}, \mu_0\right) > k\right\}$$

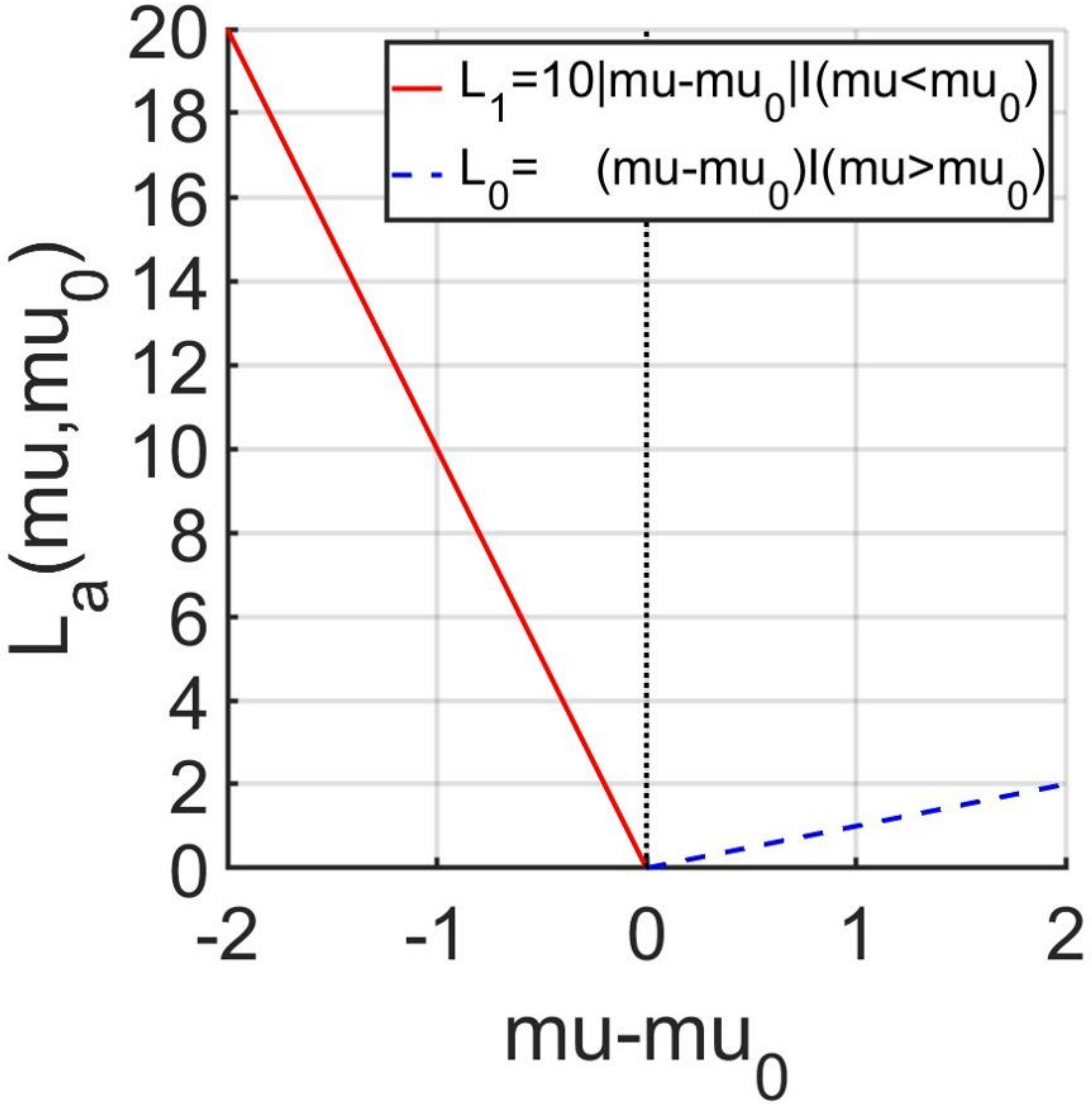
Note $W\left(\underline{\boldsymbol{y}}, \mu_0\right) = \sup_k \left\{k: W\left(\underline{\boldsymbol{y}}, \mu_0\right) > k\right\}$

= supremum of the values of k for which Bayes rule accepts $A: \mu > \mu_0$.

Interpretation

- If $W\left(\underline{\mathbf{y}}, \mu_0\right) = \mathbf{100}$, then the data support accepting A when
  - **type 1 error is no more than 100 times more serious than type 2 error**, where
  - type 1 error = falsely accepting $A: \mu > \mu_0$
  - type 2 error = falsely not accepting $A: \mu > \mu_0$
- $W\left(\underline{\mathbf{y}}, \mu_0\right)$ measures the evidence against H and in favor of A on the scale of
  - **the trade-off between the two potential decision errors of**
  - falsely accepting A (type 1) and falsely not accepting A (type 2).
- In contrast, the $\boldsymbol{p}$ **value** measures evidence against H, but not on the scale of the trade-off between the potential decision errors.
- Thus, for decision making, the p value is arguably less interpretable than $W\left(\underline{\mathbf{y}}, \mu_0\right)$.

Linear Loss Function: $l(u, v) = u - v$



Bayes Rule Critical Region under Linear Loss

$$\tilde{R} = \left\{ \underline{\boldsymbol{y}} : W\left(\underline{\boldsymbol{y}}, \mu_0\right) > k \right\}$$

where

$$W\left(\underline{\boldsymbol{y}}, \mu_0\right) = \frac{\int_{\mu_0}^{\infty} (\mu - \mu_0)\pi\left(\mu | \underline{\boldsymbol{y}}\right) d\mu}{\int_{-\infty}^{\mu_0} (\mu_0 - \mu)\pi\left(\mu | \underline{\boldsymbol{y}}\right) d\mu} = \textbf{weighted posterior odds of } \boldsymbol{A}$$

$$\pi\left(\mu | \underline{\boldsymbol{y}}\right) = \text{posterior of } \mu | \underline{\boldsymbol{y}}$$

Normal Data; Diffuse Prior

Data

$$\underline{\boldsymbol{y}} = (y_1, y_2, \ldots y_n)\; y_i \overset{iid}{\sim} N(\mu, \tau^{-1}), \tau = \sigma^{-2}$$

$$\bar{y} = n^{-1}\Sigma y_i \quad \sim N\left(\mu, \tau_{\bar{y}}^{-1}\right), \tau_{\bar{y}} = n\tau$$

$$s^2 = f^{-1}\Sigma(y_i - \bar{y})^2 \sim \Gamma\left(\frac{f}{2}, \frac{f\tau}{2}\right), f = n - 1$$

Prior (Box-Tiao)

$$\pi(\mu, \tau) \propto \tau^{-1}$$

Normal Data; Box-Tiao Prior

Posterior

$$\pi\left(\mu,\tau|\underline{\boldsymbol{y}}\right)=\pi\left(\mu|\tau,\underline{\boldsymbol{y}}\right)\pi\left(\tau|\underline{\boldsymbol{y}}\right)$$

$$\mu|\tau,\underline{\boldsymbol{y}}\sim N\left(\overline{y},\tau_{\overline{y}}^{-1}\right)$$

$$\tau|\underline{\boldsymbol{y}}\sim\Gamma\left(\frac{f}{2},\frac{fs^{2}}{2}\right)$$

Bayes Rule Critical Region

$$\tilde{R} = \left\{ \underline{\boldsymbol{y}} : W\left(\underline{\boldsymbol{y}}, \mu_0\right) > k \right\}$$

$$W\left(\underline{\boldsymbol{y}}, \mu_0\right) = M(t, f)/M(-t, f)$$

$$t = \sqrt{n}(\overline{y} - \mu_0)/s$$

$$f = n - 1$$

$$M(t, f) = (f - 1)(t^2 + f)g(z, f) + tG(z, f)$$

$g(\bullet, \nu)$ = density of Student-t with ν dof

$G(\bullet, \nu)$ = cumulative dist’n of Student-t with ν dof

Bayes Rule Critical Region, τ known

$$\tilde{R} = \left\{ \underline{\boldsymbol{y}} : W\left(\underline{\boldsymbol{y}}, \mu_0\right) > k \right\}$$

$$W\left(\underline{\boldsymbol{y}}, \mu_0\right) = M(z)/M(-z)$$

$$z = \sqrt{n}(\bar{y} - \mu_0)/\sigma$$

$$M(z) = \varphi(z) + z\Phi(z) = \lim_{f \to \infty} M(t, f)$$

$\varphi(\bullet)$ = density of $N(0,1)$ variable

$\Phi(\bullet)$ = cumulative dist'n of $N(0,1)$ variable

Interval Estimation

- $\boldsymbol{k}$**CI**$= (\tilde{\boldsymbol{\mu}}_L, \tilde{\boldsymbol{\mu}}_U)$ = values of μ_0 for which neither $A: \mu > \mu_0$ nor $A': \mu < \mu_0$ can be accepted based on the hypothesis tests of $H: \mu \leq \mu_0$ and $H': \mu \geq \mu_0$.

- When $k > 1$ is the same for both hypothesis tests,

$$\tilde{\mu}_L \text{ satisfies } W\left(\underline{\boldsymbol{y}}, \tilde{\mu}_L\right) = k$$

$$\tilde{\mu}_U \text{ satisfies } W\left(\underline{\boldsymbol{y}}, \tilde{\mu}_U\right) = k^{-1}$$

- For normal data, linear losses, and diffuse Box-Tiao prior,

$$k\text{CI} = \bar{y} \pm t_k s/\sqrt{n}, \text{ where } t_k \text{ satisfies } \frac{M(t_k, f)}{M(-t_k, f)} = k$$

Numerical Examples

$\bar{y}$	s^2	f	t	p val	$W(\underline{y}, \mu_0)$	95% CI	kCI
1.66	1	9	2.08	0.068	100.0	0.94,2.37	1.00,2.31
2.26	1	9	2.26	0.050	146.4	1.00,2.43	1.06,2.37
1.54	1	∞	1.72	0.085	100.0	0.92,2.16	1.00,2.09
1.62	1	∞	1.96	0.050	208.5	1.00,2.24	1.08,2.16

$\bar{y} = n^{-1}\Sigma y_i, s^2 = f^{-1}\Sigma(y_i - \bar{y})^2, f = n - 1$ or $\infty, t = \sqrt{n}(\bar{y} - \mu_0)/s,$

kCI$= \bar{y} \pm t_k s/\sqrt{n}$, where t_k satisfies $\frac{M(t_k, f)}{M(-t_k, f)} = k$

Two-Sample Comparison of Means

FDA

Data

$$y_{gi} \sim N\left(\mu_g, \tau^{-1}\right), \tau = \sigma^{-2}$$

Prior (Box-Tiao)

$$\mu_g \sim N\left(\mu_0, \tau_\mu^{-1}\right)$$

$$\pi\left(\mu_0, \mu_1 \tau, \tau_\mu\right) \propto \tau\left(\tau^{-1} + 2\tau_\mu^{-1}\right)^{-1}$$

$$= \tau_\mu\left(\tau_\mu + 2\tau\right)^{-1}$$

Hypothesis Test

$$\delta = \mu_2 - \mu_1$$

$$H: \delta \leq \delta_0$$

$$A: \delta > \delta_0$$

Bayes Rule Critical Region, Linear Loss

$$\tilde{R} = \{\underline{\boldsymbol{y}} : W(\underline{\boldsymbol{y}}; \delta_0) > k\}$$

$$W(\underline{\boldsymbol{y}}; \delta_0) = W(t, f; \delta_0)$$

$$t = (\bar{d} - \delta_0)/s_{\bar{d}}$$

$$s_{\bar{d}}^2 = 2s^2/n$$

$$s^2 = f^{-1}\Sigma\Sigma(y_{ai} - \bar{y}_a)^2$$

$$f = n - 1$$

Weighted Posterior Odds, Linear Loss

$$W(t, f; \delta_0) = \sup_k \{k: W(t, f; \delta_0) > k\}$$

$=$ supremum of the values of k at which $A: \delta > \delta_0$ is accepted by Bayes rule.

- If $W(t, f; \delta_0) = \mathbf{100}$, then the data support accepting A when
  - **type 1 error is no more than <u>100 times more serious</u> than type 2 error**, where
  - type 1 error = falsely accepting $A: \delta > \delta_0$
  - type 2 error = falsely not accepting $A: \delta > \delta_0$

Numerical Examples

$t_\gamma(f) =$ γth quantile of Student-t with f dof

$$t_{0.975}(\infty) = 1.96\text{: } W(1.96, \infty; \delta_0) \cong 100$$

$$t_{0.95}(\infty) = 1.645\ W(1.645, \infty; \delta_0) \cong 50$$

$$t_{0.995}(\infty) = 2.576\ W(2.576, \infty; \delta_0) \cong 500$$

Interval Estimation

- $\boldsymbol{k}\mathbf{CI} = \left(\tilde{\delta}_L\,, \tilde{\delta}_U\right) =$ values of μ_0 for which neither $A\!:\delta > \delta_0$ nor $A'\!:\delta < \delta_0$ can be accepted based on the hypothesis tests of $H\!:\delta \leq \delta_0$ and $H'\!:\delta \geq \delta_0$.

- When $k > 1$ is the same for both hypothesis tests,

$$\tilde{\delta}_L \text{ satisfies } W\left(t, f; \tilde{\delta}_L\right) = k$$

$$\tilde{\delta}_U \text{ satisfies } W\left(t, f; \tilde{\delta}_U\right) = k^{-1}$$

- For normal data, linear losses, and diffuse Box-Tiao prior,

$$k\mathrm{CI} = \bar{d} \pm t_k\, s\sqrt{2/n}, \text{ where } t_k \text{ satisfies } W(t_k, f; \delta_0) = k$$

Squared-Error Loss for Estimation

- **Result**. Squared-error loss for estimation and linear loss for testing yield the same interval estimate $(\tilde{\delta}_L, \tilde{\delta}_U)$:
  - $\tilde{\delta}_L$ satisfies $W(\underline{\boldsymbol{y}}; \tilde{\delta}_L) = k$
  - $\tilde{\delta}_U$ satisfies $W(\underline{\boldsymbol{y}}; \tilde{\delta}_U) = k^{-1}$

Squared Error Loss $L(e,\delta) = \begin{cases} k(e-\delta)^2, e \geq \delta \\ (\delta - e)^2, e < \delta \end{cases}$

Posterior Expected Loss $E[L(e,\delta)] = \int_{-\infty}^{e} k(e-\delta)^2 \pi(\delta|\underline{y})\, d\delta + \int_{e}^{\infty} (\delta - e)^2 \pi(\delta|\underline{y})\, d\delta$

$$\frac{\partial}{\partial e} E[L(e,\delta)] \overset{\text{Leibnitz}}{=} 2k \int_{-\infty}^{e} (e-\delta)\pi(\delta|\underline{y})\, d\delta - 2\int_{e}^{\infty} (\delta - e)\pi(\delta|\underline{y})\, d\delta \overset{\text{set}}{=} 0$$

$$\therefore\ k = \int_{e}^{\infty} (\delta - e)\pi(\delta|\underline{y})\, d\delta \Big/ \int_{-\infty}^{e} (e-\delta)\pi(\delta|\underline{y})\, d\delta \equiv W(\underline{\boldsymbol{y}}; \delta_0)$$

Dixon DO, Duncan DB. Minimum Bayes Risk *t* -Intervals for Multiple Comparisons, *JASA* 1975; 70 (352), 822-831.
Dixon DO. Interval Estimates Derived from Bayes Testing Rules. *JASA* 1976; 71 (354), 406-408.

Discussion

- **Summaries of evidence for a hypothesis include**
  - the $\boldsymbol{p}$ **value**, the smallest allowed type 1 error level for accepting a hypothesis,
  - **weighted posterior odds (WPO)**, the largest allowed type-1-to-type-2-error-seriousness ratio (k) for accepting a hypothesis.
- **Arguably, WPO is of more interest to decision makers than the $\boldsymbol{p}$ value.**
  - Decision makers make value judgments about the consequent benefits and harms of actions that would be taken if a hypothesis were accepted as true or not.
  - Likewise, WPO summarizes evidence for a hypothesis in terms of relative consequences of decision errors, where "consequence"= "seriousness", "importance", "harm", "risk", "loss"...
- **For stakeholders** (e.g., regulator, payer, provider, policy maker, manufacturer, doctor, patient), **the observed WPO in a study**
  - may be compared with their own value judgment of k.
  - should facilitate better discussion as to whether to accept a hypothesis as true or not.

Discussion *(continued)*

- **Proposal:** Set preliminary value of k by FDA regulatory pathway, e.g.,

Product	k_1	k_2	k_3	k_4
Drug / Biologic	Full approval	Accelerated approval	Orphan	
		Biosimilar		
Medical Device	Class III	Class II	Humanitarian	Class I
	Breakthrough	Lab developed test	EUA	Wellness
			Wearable	

$$k_1 > k_2 > k_3 > k_4 \gg 1$$

Discussion (*continued*)

- **The k-ratio method** was developed by **David B. Duncan** and his students for Bayes rule **multiple comparisons of means (MCM)** problems assuming
  - (1) **additive linear losses** of the component comparisons,
  - (2) **prior exchangeability** of the means

- **MCM Problems**
  - Comparisons of Means in a 1-Way Array (Waller)
    - $\mathrm{W}\left(t_{ij,}, F, q, f; \delta_0\right) =$ WPO for $\delta_{ij} > \delta_0$
  - Comparisons of Treatments with a Control (Brant)
    - $\mathrm{W}(t_{i0}, F_T, F_G, q_T, q_G, f; \delta_0) =$ WPO for $\delta_{i0} > \delta_0$ $(q_G = 1)$
  - Largest Mean Problem (Ranking and Selection) (Bland)
  - Comparisons of Means in a 2-Way Array (Pennello)
    - $\mathrm{W}\left(t_{ij,k}, t_{ij\bullet}, F_A, F_B, F_C, q_A, q_B, q_C, f; \delta_0\right) =$ WPO for $\delta_{ij,k} > \delta_0$

Discussion *(continued)*

Extra Results. If distance function $l(u, v)$ is constant when $u > v$ and 0 otherwise, then

(1) $W\left(\underline{y}, \mu_0\right) = \boldsymbol{\pi_A}(\mathbf{1} - \boldsymbol{\pi_A})^{-\mathbf{1}}$ is the posterior odds of $A\colon \mu > \mu_0$, where $\boldsymbol{\pi_A} = \mathbf{Pr}\left(\mu > \mu_0 | \underline{y}\right)$,

(2) specifying k is the same as specifying the posterior probability of $\boldsymbol{\pi_A} = k/(1 + k)$ at which A is accepted.

(3) in the one sample case with τ known and $\pi(\mu) \propto 1$, $W\left(\underline{\boldsymbol{y}}, \mu_0\right) =$ Bayes factor because then $\pi\left(\mu|\underline{\boldsymbol{y}}\right) = f\left(\underline{\boldsymbol{y}}|\mu\right)$ numerically.

& Device

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Questions?

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